Supersymmetric Chern-Simons Theory in Presence of a Boundary in the Light-Like Direction

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Abstract

In this paper, we will analyze a three dimensional supersymmetric Chern-Simons theory on a manifold with a boundary. The boundary we will consider in this paper will be defined by $n \cdot x = 0$, where n is a light-like vector. It will be demonstrated that this boundary is preserved under the action of the $SIM(1)$ subgroup of the Lorentz group. Furthermore, the presence of this boundary will break half of the supersymmetry of the original theory. As the original Chern-Simons theory had $\mathcal{N} = 1$ supersymmetry in absence of a boundary, it will only have $\mathcal{N} = 1/2$ supersymmetry in presence of this boundary. We will also observe that the Chern-Simons theory can be made gauge invariant by introducing new degrees of freedom on the boundary. The gauge transformation of these new degrees of freedom will exactly cancel the boundary term obtained from the gauge transformation of the Chern-Simons theory.

1 Introduction

It is known that the action for most renormalizable quantum field theories is at most quadratic in derivatives. This also includes the supersymmetric quantum field theories. So, the supersymmetric transformation of the action for these theories is expected to produce a total derivative term, apart from the bulk term. The bulk term vanishes due to the equations of motion, and the total derivative term vanishes in absence of a boundary. However, if a boundary is present, this total derivative term will give rise to a boundary contribution. Thus, the presence of a boundary is expected to break the supersymmetry of the theory. In fact, as the presence of a boundary breaks the translational invariance of the theory, and the translation invariance of the theory is related to the invariance of the theory under supersymmetry [1], it is expected that the supersymmetry will be broken due to the presence of a boundary. It is possible to

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impose suitable boundary conditions such that the supersymmetry of the theory will not be broken [2]-[3]. These boundary conditions are imposed on the Euler-Lagrange field equations. The surface terms vanish on-shell, after these boundary conditions are imposed, and this preserves the on-shell supersymmetry of the theory. However, the boundary conditions imposed on the Euler-Lagrange field equations can not be used to preserve the off-shell supersymmetry of the theory. It may be noted that various boundary conditions for supersymmetric theories have been analyzed [4]-[8]. The path integral formalism is used to quantize most supersymmetric theories, and this formalism uses off-shell fields. So, it is important to have a formalism which preserves the off-shell supersymmetry in presence of a boundary.

Such a formalism has been constructed, and in this formalism the half of the original supersymmetry is preserved off-shell. This formalism is based on modifying the original action by adding a boundary contribution to it. The supersymmetric variation of the boundary contribution exactly cancels the supersymmetric variation of the bulk theory. However, this can only be done for half the supercharges of the original theory. Hence, only half the supersymmetry of the original theory is preserved. This formalism has been used for three dimensional theories with $\mathcal{N} = 1$ supersymmetry [9]-[11]. This three dimensional formalism has been used for analyzing a system of multiple M2-branes ending on a M5-brane [12]. The action for multiple M2-branes is dual to the supergravity on $AdS_4 \times S_7$, and the $OSp(8|4)$ symmetry of the eleven dimensional supergravity on $AdS_4 \times S_7$ is realized as $\mathcal{N} = 8$ supersymmetry of this dual field theory. Furthermore, all the on-shell degrees of freedom of this theory are exhausted by the matter fields, so the gauge sector has to be described by a topological theory. It has been possible to construct such a theory which is a matter-Chern-Simons theory called the Aharony-Bergman-Jafferis-Maldacena (ABJM) theory [13]-[16]. Even though this theory only has $\mathcal{N} = 6$ supersymmetry, it is expected that its supersymmetry can be enhanced to full $\mathcal{N} = 8$ supersymmetry [17]-[18]. In fact, it coincides with a theory called the Bagger–Lambert–Gustavsson (BLG) theory for two M2-branes, and the BLG theory has $\mathcal{N} = 8$ supersymmetry [19]-[21].

The action for the matter sector of this theory is gauge invariant even in presence of a boundary. The action for the gauge sector of this theory is described by a Chern-Simons theory. It is well known that the gauge transformation of a Chern-Simons theory produces a surface term. So, in the presence of a boundary the gauge transformation of the ABJM theory generates a nonvanishing boundary term. However, it has been demonstrated that if new boundary degrees of freedom are introduced, then gauge invariance of the ABJM theory can be restored even in presence of a boundary [22]-[24]. This is because the gauge transformation of these new boundary degrees of freedom exactly cancels the boundary contribution generated from the gauge transformation of the bulk action. It is important to study the open multiple M2-brane action as it can be used to understand the physics of M5-branes. It may be noted that a system of M2-branes intersecting with M5-branes has been studied using a fuzzy funnel solution [25]-[30]. A system of multiple M2-branes ending on two M9-branes is expected to generate $E_8 \times E_8$ symmetry from the gravitational anomaly [31]-[32]. The BLG theory has been used to study novel quantum

geometry on the M5-brane world-volume by analyzing M2-branes ending on M5-branes with a constant C-field [33], and the BLG action with Nambu-Poisson 3-bracket has been identified as the M5-brane action with a large world-volume C-field [34]. The boundary Chern-Simons theory has many other possible applications. It is possible for D-branes to end on other objects in string theory [35]-[39]. The Chern-Simons theory has been used for analyzing a system of open strings ending on D-branes in the the A-model topological string theory [40], and the Holomorphic Chern–Simons theory has been used for analyzing the B-model in the string theory [41]. Thus, it is important to study the Chern-Simons theory in presence of a boundary.

It may be noted that the gauge and supersymmetric invariance of Chern-Simons-matter theories has already been studied for boundaries in the space-like direction [9]-[12]. However, such an analysis has not been performed for the boundaries in the light-like direction. It might be possible to generalize the formalism developed to preseve the gauge and supersymmetry of the Chern-Simons-matter theories in presence of a boundary along a space-like direction to preserve these symmetries for the Chern-Simons-matter theories in presence of a boundary along a light-like direction. Such an generalization will have to take into consideration the crucial differences between boundaries along space-like and light-like directions. For a boundary along a space-like direction the pullback of the metric on the boundary has rank two, but for a boundary along a light-like direction the pullback of the metric has only rank one. However, it is possible to use some additional structure that occurs only for a boundary along a light-like direction to construct a gauge and supersymmetric invariant Chern-Simons theory in presence of a boundary along the light-like direction. Even thought the Lorentz symmetry breaks for boundaries along both the space-like and light-like directions, the boundaries along the light-like direction preserves a sub-group of the Lorentz group. A boundary in a light-like direction, preserves the $SIM(1)$ group of the spacetime. Thus, we will use the $SIM(1)$ superspace formalism [42] to describe Chern-Simons theory in presence of a boundary along a light-like direction. It has been demonstrated that half the supersymmetry of the Lorentz invariant theory can be retained, when the Lorentz symmetry is broken down to the $SIM(1)$ symmetry. This is done without adding additional boundary terms to the original action. The advantage of this $SIM(1)$ superspace formalism is that the one-loop effective action for various theories can be easily calculated using $SIM(1)$ superspace formalism. In fact, one-loop effective action for a Wess-Zumino model has been calculated using $SIM(2)$ superspace formalism [44]. The calculation of such effective action for Chern-Simons theories even in presence a boundary is non-trivial using the methods developed for analysing space-like boundaries. It may be noted that we do not need to add additional boundary terms to preserve half the supersymmetry of a Chern-Simons theory in presence of a boundary along light-like direction, if we use the $SIM(1)$ superspace formalism. However, for Chern-Simons theories, we have to add additional boundary terms to preserve gauge symmetry, even in $SIM(1)$ superspace formalism.

2 Chern-Simons Theory

The gauge covariant derivatives

$$\nabla_\alpha = D_\alpha - i\Gamma_\alpha, \qquad \nabla_{\alpha\beta} = \partial_{\alpha\beta} - i\Gamma_{\alpha\beta}, \tag{1}$$

are expressed with the help of connections Γ_α, $\Gamma_{\alpha\beta}$, where the spinor derivatives satisfy

$$\{D_\alpha, D_\beta\} = -2\partial_{\alpha\beta}. \tag{2}$$

Sometimes we will use Latin uppercase indices $A, B, \ldots$ to represent both spinor $A = \alpha$ and vector $A = \alpha\beta$ indices. In this notation we write (1) as

$$\nabla_A = \mathcal{D}_A - i\Gamma_A, \tag{3}$$

where the derivatives are $\mathcal{D}_A = (\mathcal{D}_\alpha, \mathcal{D}_{\alpha\beta}) = (D_\alpha, \partial_{\alpha\beta})$. It is also useful to assign Grassmann parity to indices. A spinor index $A = \alpha$ will be Grassmann odd $\widetilde{A} = 1$ and a vector index $A = \alpha\beta$ will be Grassmann even $\widetilde{A} = 0$.

The (anti)commutators among gauge covariant derivatives are

$$\begin{aligned} \{\nabla_\alpha, \nabla_\beta\} &= -2\nabla_{\alpha\beta}, \qquad [\nabla_\alpha, \nabla_{\beta\gamma}] = C_{\alpha(\beta}W_{\gamma)}, \\ [\nabla_{\alpha\beta}, \nabla_{\gamma\delta}] &= -\frac{1}{2}C_{\alpha\gamma}F_{\beta\delta} - \frac{1}{2}C_{\alpha\delta}F_{\beta\gamma} - \frac{1}{2}C_{\beta\delta}F_{\alpha\gamma} - \frac{1}{2}C_{\beta\gamma}F_{\alpha\delta}, \end{aligned} \tag{4}$$

where the field strengths are

$$\begin{aligned} W_\alpha &= -\frac{i}{2}D^\beta D_\alpha \Gamma_\beta - \frac{1}{2}[\Gamma^\beta, D_\beta \Gamma_\alpha] + \frac{i}{6}[\Gamma^\beta, \{\Gamma_\beta, \Gamma_\alpha\}], \qquad \nabla^\alpha W_\alpha = 0, \\ \Gamma_{\alpha\beta} &= -\frac{1}{2}\left(D_{(\alpha}\Gamma_{\beta)} - i\{\Gamma_\alpha, \Gamma_\beta\}\right), \qquad F_{\alpha\beta} = \frac{1}{2}\nabla_{(\alpha}W_{\beta)}. \end{aligned} \tag{5}$$

The connections are subject to the gauge transformation

$$\Gamma_A^{(K)} = e^{iK}\Gamma_A e^{-iK} + ie^{iK}(\mathcal{D}_A e^{-iK}), \tag{6}$$

where K is a scalar superfield. The infinitesimal version of the above gauge transformations is

$$\delta_g^{(K)}\Gamma_A = i[K, \Gamma_A] + \mathcal{D}_A K. \tag{7}$$

The $\mathcal{N} = 1$ Chern-Simons action is

$$S^{\text{cs}}[\Gamma_A] = \frac{k}{4\pi}\text{tr}\int \mathrm{d}^3x\mathrm{d}^2\theta \left(\Gamma^\alpha W_\alpha - \frac{1}{6}\{\Gamma^\alpha, \Gamma^\beta\}\Gamma_{\alpha\beta}\right), \tag{8}$$

where k is the level of the Chern-Simons theory. The gauge transformations of the Chern-Simons theory give rise to a surface term. This surface term does not cause any troubles for a theory without a boundary, but breaks the gauge invariance if a boundary is present. The infinitesimal gauge transformation (7) gives the surface term

$$\delta_g^{(K)}S^{\text{cs}}[\Gamma_A] = \frac{k}{4\pi}\text{tr}\int \mathrm{d}^3x\mathrm{d}^2\theta \left(D^\alpha\left(KW_\alpha - \frac{1}{3}K[\Gamma^\beta, \Gamma_{\alpha\beta}]\right) - \frac{1}{6}\partial_{\alpha\beta}\left(K\{\Gamma^\alpha, \Gamma^\beta\}\right)\right). \tag{9}$$

3 $SIM(1)$ Supersymmetry

The detailed derivation of the $SIM(1)$ supersymmetry can be found in [42], here we are going to just review some basic facts that we are going to use in this paper. The $SIM(1)$ group is a subgroup of the Lorentz group that preserves a given light-like direction, this means that there is a light-like vector n which is preserved up to a rescaling by the action of the $SIM(1)$ group. This condition can also be formulated in the language of the double cover group $SL(2,\mathbb{R})$ of the Lorentz group $SO_+(2,1)$. The light-like vector n can be written as $n^{\alpha\beta} = \xi^\alpha \xi^\beta$, where the commuting spinor ξ is determined uniquely up to a sign. The $SIM(1)$ group is a subgroup of $SO_+(2,1)$ that preserves the light-like vector n up to a rescaling. This corresponds to a subgroup of $SL(2,\mathbb{R})$ determined by the condition that ξ is preserved up to a rescaling. In this paper, we will assume that ξ and n are chosen such that their nonzero components are $\xi^+ = 1$ and $n^{++} = 1$. The $SIM(1)$ transformation of a general spinor ψ can be written as

$$\begin{pmatrix} \psi'^+ \\ \psi'^- \end{pmatrix} = \begin{pmatrix} e^{-A} & -B \\ 0 & e^{A} \end{pmatrix} \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \qquad \Leftrightarrow \qquad \begin{pmatrix} \psi'_+ \\ \psi'_- \end{pmatrix} = \begin{pmatrix} e^{A} & 0 \\ B & e^{-A} \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \tag{10}$$

where $A, B \in \mathbb{R}$. Notice, that instead of matrices from $SL(2,\mathbb{R})$ that we would use in the case of the Lorentz group $SO_+(2,1)$, we use only subgroup of $SL(2,\mathbb{R})$ consisting of triangular matrices.

When the symmetry is reduced to the $SIM(1)$ subgroup of the Lorentz group, the space of spinors $\mathcal{S}$ do not constitute an irreducible representation. While the group $SL(2,\mathbb{R})$ is semisimple and all representations can be written as a sum of irreducible representations, the group $SIM(1)$ is solvable and not all representations can be written as a sum of irreducible representations. One such representations is the one that we have on $\mathcal{S}$. The subspace $\mathcal{S}_{\text{invariant}}$ consisting of all spinors that satisfy the condition $\not{n}\psi = 0$ is irreducible, and it is the only irreducible subspace of $\mathcal{S}$. However, we have an irreducible representation on the quotient space $\mathcal{S}_{\text{quotient}} = \mathcal{S}/\mathcal{S}_{\text{invariant}}$. In our choice of n, the space $\mathcal{S}_{\text{invariant}}$ consists of spinors for which the ψ_+ coordinate vanishes, the space $\mathcal{S}_{\text{invariant}}$ can be conveniently described if we choose in each equivalence class a representative which has the coordinate ψ_- equal to zero. The infinitesimal $SIM(1)$ transformations are

$$\begin{pmatrix} 0 \\ \psi'_- \end{pmatrix} = e^{-A} \begin{pmatrix} 0 \\ \psi_- \end{pmatrix}, \qquad \left[\begin{pmatrix} \psi'_+ \\ 0 \end{pmatrix}\right] = e^{A} \left[\begin{pmatrix} \psi_+ \\ 0 \end{pmatrix}\right]. \tag{11}$$

The $SIM(1)$ supersymmetry is not the symmetry that we get directly from super-Poincare symmetry when the Lorentz symmetry is broken down to the $SIM(1)$ symmetry, we also have to break half of the supersymmetry of the theory. Thus, the $\mathcal{N} = 1$ supersymmetry is reduced to $\mathcal{N} = 1/2$ supersymmetry. The part of supersymmetry that we keep corresponds to supersymmetry transformations generated by ϵQ, with the infinitesimal anticommuting parameter satisfying $\not{n}\epsilon = 0$.

The number of anticommuting coordinates parameterizing $SIM(1)$ superspace is half of the number of coordinates that parametrize the original $\mathcal{N} = 1$ superspace. Thus, the $SIM(1)$ supersymmetry only contains a single supercharge S_+, and there is only one

anticommuting coordinate θ_- parameterizing $SIM(1)$ superspace. This supercharge corresponds to the spinor derivative d_+. Thus, the generator of the $SIM(1)$ supersymmetry and the corresponding spinor derivative are given by

$$S_+ = \partial_+ + i\theta_-\partial_{++}, \qquad\qquad d_+ = \partial_+ - i\theta_-\partial_{++}. \tag{12}$$

They satisfy

$$\{S_+, S_+\} = 2\partial_{++}, \qquad \{S_+, d_+\} = 0, \qquad \{d_+, d_+\} = -2\partial_{++}, \qquad \partial_+\theta_- = -i. \tag{13}$$

It may be noted that the anticommuting coordinate θ_- transforms under the $SIM(1)$ group as a spinor from $\mathcal{S}_{\text{invariant}}$. The spinor derivative and the generator of the supersymmetry transform under the $SIM(1)$ group as spinors from $\mathcal{S}_{\text{quotient}}$.

4 Boundary Supersymmetry

In this section, we are going to investigate how the symmetry of a theory is reduced, if we assume that there is a boundary consisting of points that satisfy the condition $n \cdot x = 0$. We are going to show that the $SIM(1)$ supersymmetry arises naturally in this context [42]. We will review this discussion here, because it demonstrates which surface terms are relevant for this boundary theory.

In our particular choice of n the condition $n \cdot x = 0$ means that $x^{--} = 0$. This set of points is preserved under the action of the $SIM(1)$ group, because the direction of n is preserved. We can also perform shifts in x^{++} and x^{+-} directions that are generated by P_{+-}, P_{--}. The shift in the x^{--} direction does not preserve the boundary, thus the P_{--} generator cannot be part of the symmetry group.

In order to determine which part of supersymmetry is preserved, we will assume that there is a scalar superfield Φ which is constrained by the condition that it vanishes on the boundary. Such superfield may appear for example in a matter Chern-Simons theory. The amount of unbroken supersymmetry will follow from the requirement that the boundary condition

$$\Phi|_{x^{--}=0} = 0, \tag{14}$$

is invariant. The infinitesimal supersymmetry transformation changes this boundary condition as

$$\begin{aligned}
\delta\Phi|_{x^{--}=0} &= -(\epsilon^\alpha Q_\alpha \Phi)|_{x^{--}=0} \\
&= -\left[\epsilon^+(\partial_+ + \theta^+\partial_{++} + \theta^-\partial_{+-})\Phi + \epsilon^-(\partial_- + \theta^+\partial_{+-} + \theta^-\partial_{--})\Phi\right]|_{x^{--}=0} \\
&= -\epsilon^-(\theta^-\partial_{--}\Phi)|_{x^{--}=0}.
\end{aligned} \tag{15}$$

This result clearly shows that the boundary condition (14) is left unchanged only if the infinitesimal parameter ϵ satisfies the condition $\not{n}\epsilon = \epsilon^- = 0$. This is the same condition that we used to break down the $\mathcal{N} = 1$ Lorentz supersymmetry down to the $\mathcal{N} = 1/2$ $SIM(1)$ supersymmetry.

The only difference between the symmetry that we have just described and the $SIM(1)$ supersymmetry from the previous section is that P_{--} is not part of the boundary supersymmetry algebra. This does not affect most of results that we have in the $SIM(1)$ supersymmetry. It should also be clear that only the surface term which is a total ∂_{--} derivative will be relevant when we will investigate the gauge invariance of the Chern-Simons action.

5 Chern-Simons Theory in the $SIM(1)$ Superspace

In order to write down the Chern-Simons theory in the $SIM(1)$ superspace we introduce the projections

$$\gamma_A = \Gamma_A|_{\theta_+=0}, \tag{16}$$

of connections $\Gamma_\alpha, \Gamma_{\alpha\beta}$ [45]. The projection $|_{\theta_+=0}$ removes the dependence on the anticommuting coordinate θ_+, which does not parametrize the $SIM(1)$ superspace. The gauge transformations are

$$\gamma_A^{(K)} = e^{ik}\gamma_A e^{-ik} + ie^{ik}(\mathcal{D}_A e^{-ik}), \tag{17}$$

where k is the projection $k = K|_{\theta_+=0}$ of the scalar superfield K and the derivatives are $\mathcal{D}_A = (\mathcal{D}_+, \mathcal{D}_{\alpha\beta}) = (d_+, \partial_{\alpha\beta})$. The infinitesimal version of the above gauge transformations is

$$\delta_g^{(K)}\gamma_A = i[k, \gamma_A] + \mathcal{D}_A k. \tag{18}$$

The rules (17), (18) do not hold for γ_- because the coordinate θ^- was lost when we made projection on to the $SIM(1)$ superspace, thus we do not have anything that would correspond to D_-. Instead we define a projection $\kappa_- = (D_- K)|_{\theta_+=0}$ and instead of (18) we have the infinitesimal gauge transformation

$$\delta_g^{(K)}\gamma_- = i[k, \gamma_-] + \kappa_-. \tag{19}$$

The projections γ_+, γ_-, γ_{++}, γ_{+-}, γ_{--} provide all information that we need to describe the gauge theory. There is only one constraint that they have to satisfy

$$d_+\gamma_+ = -\gamma_{++} + \frac{i}{2}\{\gamma_+, \gamma_+\}. \tag{20}$$

The infinitesimal $SIM(1)$ transformation change a spinor ψ, according to (10), as

$$\delta_s\psi_+ = A\psi_+, \qquad \delta_s\psi_- = -A\psi_- + B\psi_+. \tag{21}$$

In the case of superfields, the infinitesimal change is calculated by applying the above rules on each index that it carries plus the change caused by the shift in superspace coordinates.

The Chern-Simons action can be written as a sum of the bulk part $S^{\mathrm{cs}}_{\mathrm{bulk}}$, which contains the part that can be written as an integral over the whole space-time, and the

boundary part $S^{\text{cs}}_{\text{boundary}}$, which contains the part that can be written as a total ∂_{--} derivative and is nonvanishing only on the boundary

$$S^{\text{cs}}[\gamma_A] = S^{\text{cs}}_{\text{bulk}}[\gamma_A] + S^{\text{cs}}_{\text{boundary}}[\gamma_A]. \tag{22}$$

We will assume that the space-time is infinite in directions tangent to the boundary, we will not keep track of terms that can be written as total derivatives in these directions. The Chern-Simons theory on a manifold without a boundary in $SIM(1)$ superspace has already been discussed in [45]. The action that has been obtained corresponds to $S^{\text{cs}}_{\text{bulk}}$, and it is given by

$$S^{\text{cs}}_{\text{bulk}}[\gamma_A] = \frac{k}{4\pi}\text{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\Big(2\gamma_{+-}w_- + \gamma_+ f_{--} - \gamma_{--}w_+ - i\gamma_+[\gamma_{+-},\gamma_{--}]\Big). \tag{23}$$

Because there was no boundary, the boundary part of the action $S^{\text{cs}}_{\text{boundary}}$ was not discussed. The the boundary action can be easily found if we look at the derivation of $S^{\text{cs}}_{\text{bulk}}$ in [45]. The only place where a total ∂_{--} derivative appeared was in the identity

$$\text{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\left[\partial_{--}(\{\gamma_+,\gamma_+\}\gamma_-)\right] = \text{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\left[2\{\gamma_+,\gamma_-\}(\partial_{--}\gamma_+) + \{\gamma_+,\gamma_+\}(\partial_{--}\gamma_-)\right]. \tag{24}$$

The appropriate multiple of the left side gives the boundary part of the action, which is

$$S^{\text{cs}}_{\text{boundary}}[\gamma_A] = \frac{k}{4\pi}\text{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big(-\frac{i}{6}\{\gamma_+,\gamma_+\}\gamma_-\Big). \tag{25}$$

The projections $w_\alpha = W_\alpha|_{\theta_+=0}$ and $f_{\alpha\beta} = F_{\alpha\beta}|_{\theta_+=0}$ can be calculated as

$$\begin{aligned}
w_+ &= d_+\gamma_{+-} - \partial_{+-}\gamma_+ - i[\gamma_+,\gamma_{+-}],\\
w_- &= \frac{1}{2}\left(d_+\gamma_{--} - \partial_{--}\gamma_+ - i[\gamma_+,\gamma_{--}]\right),\\
f_{++} &= -\partial_{++}\gamma_{+-} + \partial_{+-}\gamma_{++} + i[\gamma_{++},\gamma_{+-}],\\
f_{+-} &= \frac{1}{2}\left(-\partial_{++}\gamma_{--} + \partial_{--}\gamma_{++} + i[\gamma_{++},\gamma_{--}]\right),\\
f_{--} &= -\partial_{+-}\gamma_{--} + \partial_{--}\gamma_{+-} + i[\gamma_{+-},\gamma_{--}].
\end{aligned} \tag{26}$$

We should also note that neither the bulk action $S^{\text{cs}}_{\text{bulk}}$ nor the boundary action $S^{\text{cs}}_{\text{boundary}}$ are separately $SIM(1)$ invariant, if a boundary is present. The $SIM(1)$ transformation of the bulk action results in a surface term that has to be canceled by terms that we get from the $SIM(1)$ transformation of the boundary action. It can be shown that

$$\delta_s S^{\text{cs}}_{\text{bulk}}[\gamma_A] = -\delta_s S^{\text{cs}}_{\text{boundary}}[\gamma_A] = B\frac{k}{4\pi}\text{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\left(\frac{i}{6}\gamma_+\{\gamma_+,\gamma_+\}\right). \tag{27}$$

5.1 Infinitesimal Gauge Transformation

Let us look at the surface term that we get as a result of an infinitesimal gauge transformation. We will keep track only of those terms that are important for our boundary theory, that is terms that contain ∂_{--} derivative.

The infinitesimal change of $S^{\mathrm{cs}}_{\mathrm{bulk}}$ was already calculated in [45]

$$\delta_g^{(K)} S^{\mathrm{cs}}_{\mathrm{bulk}}[\gamma_A] = \frac{k}{4\pi}\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\big(- k(d_+\gamma_{+-}) + k(\partial_{+-}\gamma_+)\big), \tag{28}$$

where we kept only the surface term which is a total ∂_{--} derivative.

The infinitesimal change of the boundary term can be calculated with the help of (18) and (19). The result is

$$\begin{aligned}\delta_g^{(K)} S^{\mathrm{cs}}_{\mathrm{boundary}}[\gamma_A] = \frac{k}{4\pi}\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big(&- \frac{i}{6}\{i[k,\gamma_+] + d_+k,\gamma_+\}\gamma_- \\ &- \frac{i}{6}\{\gamma_+, i[k,\gamma_+] + d_+k\}\gamma_- - \frac{i}{6}\{\gamma_+,\gamma_+\}(i[k,\gamma_-] + \kappa_-)\Big).\end{aligned} \tag{29}$$

All terms with the commutator $i[k,\cdot]$ drop out of the calculation due to the cyclic property of the trace and the super-Jacobi identity, moreover the term $\{d_+k,\gamma_+\}\gamma_-$ appears twice, so the result could be written as

$$\delta_g^{(K)} S^{\mathrm{cs}}_{\mathrm{boundary}}[\gamma_A] = \frac{k}{4\pi}\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big(- \frac{i}{3}(d_+k)\{\gamma_+,\gamma_-\} - \frac{i}{6}\kappa_-\{\gamma_+,\gamma_+\}\Big). \tag{30}$$

5.2 Finite Gauge Transformation

The gauge transformation generated by K (or equivalently by k and κ_-) changes the action by a boundary term, which we will be denoted as S^{Δ}. We may write this as

$$S^{\mathrm{cs}}[\gamma_A^{(K)}] = S^{\mathrm{cs}}[\gamma_A] + S^{\Delta}[\gamma_A; k, \kappa_-], \tag{31}$$

where $\gamma_A^{(K)}$ are gauge transformed connections. Note that the boundary term depends on both γ_A, k and κ_-. We will state the result for S^{Δ} now, the proof will be provided later in this section. The boundary contribution is

$$\begin{aligned}S^{\Delta}[\gamma_A; k, \kappa_-] = \frac{k}{4\pi}\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big[&- \frac{i}{6}(\gamma_- + k_-)\{\gamma_+ + k_+, \gamma_+ + k_+\} + \frac{i}{6}\gamma_-\{\gamma_+,\gamma_+\} \\ &+ k_+\gamma_{+-} - \gamma_+ k_{+-} + \int_0^1 \mathrm{d}s\left(\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_+^{(s)}\right)k_{+-}^{(s)} - k_+^{(s)}\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_{+-}^{(s)}\right)\right)\Big],\end{aligned} \tag{32}$$

with $k_A^{(s)}$ and k_A defined as

$$k_A^{(s)} = i(\mathcal{D}_A e^{-isk})e^{isk}, \qquad k_A = k_A^{(s)}|_{s=1} = i(\mathcal{D}_A e^{-ik})e^{ik}. \tag{33}$$

The superfield k_- is not covered by the above definition for the same reasons as in (19). Instead, it is defined as [1]

$$k_- = i(D_- e^{-isK})e^{isK}|_{\theta_+=0}. \tag{34}$$

The boundary contribution (32) should depend only on the value of the group element e^{iK}, not on the value of superfield K used to parametrize it. This is clearly true for terms that only contain k_A and γ_A, because k_A depend only on the group element e^{ik} (and e^{iK} in the case of k_-). The uniqueness of terms with $k_A^{(s)}$ is not so clear, because there might be multiple choices of k that give the same group element e^{ik}. We are going to show that for gauge groups that are simply connected and have surjective exponential map the term with integral over s is also well defined. The group $SU(N)$ is one of the groups for which these conditions are met. The s integral can be understood as an integral along the curve $[0,1] \ni s \to g(s) = e^{isk}$ that connects the identity element with the element e^{ik}. The integrand is a 1-form

$$\omega = \mathrm{tr}\left[\mathrm{d}_g\big((d_+ g^{-1})g\big)\big((\partial_{+-} g^{-1})g\big) - \big((d_+ g^{-1})g\big)\mathrm{d}_g\big((\partial_{+-} g^{-1})g\big)\right], \tag{35}$$

where d_g denotes the exterior derivative with respect to the group element g. It is trivial to show that $\mathrm{d}_g\omega = 0$. The fact that the form ω is closed together with the assumption that the group is simply connected leads to the conclusion that the result of the integral is independent on the choice of a path which we pick to connect the identity and the element e^{ik}. We need the surjectivity requirement of the exponential map to ensure that all group elements can be written as e^{ik}. It seems that this requirement would not be necessary, if we did not assume a particular parametrization and expressed the result as a curve integral of ω.

Although it is possible to calculate S^Δ by applying gauge transformation (17), we will use a different approach. We will show that (32) is what we would get if we considered a finite gauge transformation as a series of infinitesimal ones.

Consider a gauge transformation generated by $(1+\epsilon)K$, where ϵ is an infinitesimal parameter. According to (31) we have

$$\begin{aligned} S^{\mathrm{cs}}[\gamma_A^{(K+\epsilon K)}] &= S^{\mathrm{cs}}[\gamma_A] + S^\Delta[\gamma_A; k+\epsilon k, \kappa_- + \epsilon\kappa_-] \\ &= S^{\mathrm{cs}}[\gamma_A] + S^\Delta[\gamma_A; k, \kappa_-] + \epsilon\hat{\delta}_g^{(K)} S^\Delta[\gamma_A; k, \kappa_-], \end{aligned} \tag{36}$$

where $\epsilon\hat{\delta}_g^{(K)}$ is used to denote an infinitesimal transformation that changes K to $(1+\epsilon)K$ but leaves γ_A unchanged, i.e.

$$\hat{\delta}_g^{(K)} k = k, \qquad \hat{\delta}_g^{(K)}\kappa_- = \kappa_-, \qquad \hat{\delta}_g^{(K)}\gamma_A = 0. \tag{37}$$

We may also understand the gauge transformation generated by $(1+\epsilon)K$ as a composition of a finite gauge transformation with an infinitesimal gauge transformation. Here K

[1] This could be also written as

$$k_- = \int_0^1 \mathrm{d}s\left(e^{-isk}\kappa_- e^{isk}\right).$$

parameterizes the finite gauge transformations, and ϵK parameterizes the infinitesimal gauge transformation. Thus, the alternative method to calculate the gauge transformation is

$$\begin{aligned} S^{\mathrm{cs}}[\gamma_A^{(K+\epsilon K)}] &= S^{\mathrm{cs}}[\gamma_A^{(K)}] + \epsilon \delta_g^{(K)} S^{\mathrm{cs}}[\gamma_A^{(K)}] \\ &= S^{\mathrm{cs}}[\gamma_A] + S^{\Delta}[\gamma_A; k, \kappa_-] + \epsilon \delta_g^{(K)} S^{\mathrm{cs}}[\gamma_A] + \epsilon \delta_g^{(K)} S^{\Delta}[\gamma_A; k, \kappa_-]. \end{aligned} \tag{38}$$

In this case $\delta_g^{(K)}$ changes γ_A according to (18), (19) but does not affect k and κ_-, so $\delta_g^{(K)} k = \delta_g^{(K)} \kappa_- = 0$. Comparison of terms with ϵ in (36) and (38) gives the equation

$$\hat{\delta}_g^{(K)} S^{\Delta}[\gamma_A; k, \kappa_-] - \delta_g^{(K)} S^{\Delta}[\gamma_A; k, \kappa_-] = \delta_g^{(K)} S^{\mathrm{cs}}[\gamma_A]. \tag{39}$$

In order to prove that S^{Δ} is correct, we have to show that it satisfies the above equation and the boundary condition $S^{\Delta}[\gamma_A; 0, 0] = 0$. The verification of the boundary condition is trivial, we have already calculated the right side of (39) in (28) and (30), what remains is to evaluate expressions on the left side. Before we proceed with it, we are going to derive a few useful identities. The first identity is

$$\begin{aligned} \mathcal{D}_A k_B^{(s)} - (-1)^{\widetilde{A}\widetilde{B}} \mathcal{D}_B k_A^{(s)} &= i(\mathcal{D}_A \mathcal{D}_B e^{-isk}) e^{isk} + i(-1)^{\widetilde{A}\widetilde{B}} (\mathcal{D}_B e^{-isk})(\mathcal{D}_A e^{isk}) \\ &- i(-1)^{\widetilde{A}\widetilde{B}} (\mathcal{D}_B \mathcal{D}_A e^{-isk}) e^{isk} - i(\mathcal{D}_A e^{-isk})(\mathcal{D}_B e^{isk}) = k^{(s)}_{[A,B]_\pm} - i[k_A^{(s)}, k_B^{(s)}]_\pm, \end{aligned} \tag{40}$$

where we used that $\mathcal{D}_A e^{isk} = -e^{isk}(\mathcal{D}_A e^{-isk}) e^{isk}$. The symbol $k^{(s)}_{[A,B]_\pm}$ is used to denote

$$k^{(s)}_{[A,B]_\pm} = i([\mathcal{D}_A, \mathcal{D}_B]_\pm e^{-isk}) e^{isk}. \tag{41}$$

For example, when we set $A = +$, $B = +$ we obtain the identity

$$d_+ k_+^{(s)} = \frac{1}{2}(d_+ k_+^{(s)} + d_+ k_+^{(s)}) = -k_{++}^{(s)} - \frac{i}{2}[k_+^{(s)}, k_+^{(s)}], \tag{42}$$

where we used $\{\mathcal{D}_+, \mathcal{D}_+\} = \{d_+, d_+\} = -2\partial_{++} = -2\mathcal{D}_{++}$. Another identity that can be easily derived is

$$\frac{\mathrm{d}}{\mathrm{d}s} k_A^{(s)} = \mathcal{D}_A k - i[k, k_A^{(s)}]. \tag{43}$$

We can use the fact that $\hat{\delta}_g^{(K)}(sk) = s\frac{\mathrm{d}}{\mathrm{d}s}(sk)$ to find the infinitesimal transformation of $k_A^{(s)}$

$$\hat{\delta}_g^{(K)} k_A^{(s)} = s\frac{\mathrm{d}}{\mathrm{d}s} k_A^{(s)} = s\left(\mathcal{D}_A k - i[k, k_A^{(s)}]\right). \tag{44}$$

If we set $s = 1$ in (40) we get

$$\mathcal{D}_A k_B - (-1)^{\widetilde{A}\widetilde{B}} \mathcal{D}_B k_A = k_{[A,B]_\pm} - i[k_A, k_B]_\pm. \tag{45}$$

Similar methods can be used to calculate the infinitesimal change $\hat{\delta}_g^{(K)}$ of k_A and k_-

$$\hat{\delta}_g^{(K)} k_A = \mathcal{D}_A k - i[k, k_A], \qquad \hat{\delta}_g^{(K)} k_- = \kappa_- - i[k, k_-]. \tag{46}$$

Now, we are ready to evaluate the expressions on the left side of (39). The first term in (32) does not give any contribution because the infinitesimal change of combinations $\gamma_A + k_A$ is

$$\left(\hat{\delta}_g^{(K)} - \delta_g^{(K)}\right)(\gamma_A + k_A) = -i[k, \gamma_A + k_A], \tag{47}$$

and the $-i[k, \cdot]$ commutators drop out because of the super-Jacobi identity and the cyclic property of the trace. The other terms that are outside of the s-integral give

$$\begin{aligned}\left(\hat{\delta}_g^{(K)} - \delta_g^{(K)}\right)&\left(\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big[k_+\gamma_{+-} - \gamma_+k_{+-} + \frac{i}{6}\gamma_-\{\gamma_+, \gamma_+\}\Big]\right)\\ &= \mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big[(d_+k)\gamma_{+-} - k_+(\partial_{+-}k) + (d_+k)k_{+-} - \gamma_+(\partial_{+-}k)\\ &\qquad - \frac{i}{6}\kappa_-\{\gamma_+, \gamma_+\} - \frac{i}{6}\gamma_-\{d_+k, \gamma_+\} - \frac{i}{6}\gamma_-\{\gamma_+, d_+k\}\Big].\end{aligned} \tag{48}$$

As before, all terms with $-i[k, \cdot]$ cancel among themselves and yield zero net contribution. In the next step, we integrate the first four terms by parts to move the derivatives d_+, ∂_{+-} in front of k so they act on γ_A, k_A. The expression $k(\partial_{+-}k_+ - d_+k_{+-})$ that we get from the second and the third term can be replaced with $ik[k_+, k_{+-}]$ because of the identity (45). Thus, the result for the part that is outside of the s-integral is

$$\begin{aligned}\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big[&- k(d_+\gamma_{+-}) + k(\partial_{+-}\gamma_+) + ik[k_+, k_{+-}]\\ &- \frac{i}{6}\kappa_-\{\gamma_+, \gamma_+\} - \frac{i}{3}(d_+k)\{\gamma_+, \gamma_-\}\Big].\end{aligned} \tag{49}$$

The last piece which we need to evaluate on the left side of (39) is the part of (32) that is inside the s-integral. The infinitesimal change $\delta_g^{(K)}$ does not give any contribution because $\delta_g^{(K)}k_A = 0$. The change $\hat{\delta}_g^{(K)}$ can be easily calculated if we write $\hat{\delta}_g^{(K)}k_A^{(s)}$ as $s\frac{\mathrm{d}}{\mathrm{d}s}k_A^{(s)}$. The result obtained by following this procedure can be written as

$$\begin{aligned}\hat{\delta}_g^{(K)}&\left(\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\int_0^1 \mathrm{d}s\left[\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_+^{(s)}\right)k_{+-}^{(s)} - k_+^{(s)}\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_{+-}^{(s)}\right)\right]\right)\\ &= \mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\int_0^1 \mathrm{d}s\left(\tfrac{\mathrm{d}}{\mathrm{d}s}\left[s\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_+^{(s)}\right)k_{+-}^{(s)} - sk_+^{(s)}\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_{+-}^{(s)}\right)\right]\right).\end{aligned} \tag{50}$$

The substitution for $\frac{\mathrm{d}}{\mathrm{d}s}k_A^{(s)}$ according to (43) and integration over s gives

$$\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big((d_+k)k_{+-} - i[k, k_+]k_{+-} - k_+(\partial_{+-}k) + ik_+[k, k_{+-}]\Big). \tag{51}$$

As before, we use integration by parts to move the derivatives d_+ and ∂_{+-} so they act on $k_+^{(s)}$, $k_{+-}^{(s)}$

$$\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big(k\left(-d_+k_{+-} + \partial_{+-}k_+ - 2i[k_+, k_{+-}]\right)\Big). \tag{52}$$

This could be further simplified with the identity (45), and the result is

$$\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big(-ik[k_+,k_{+-}]\Big). \tag{53}$$

The sum of (49) and (53) gives the left side of equation (39) and this is equal to the right side, which is equal to a sum of (28) and (30). This, together with fulfillment of the boundary condition $S^{\Delta}[\gamma_A;0,0]=0$, proves that (32) correctly describes change of the Chern-Simons action for finite gauge transformations.

5.3 Boundary Superfield

The Chern-Simons action is not gauge invariant, gauge transformations yield a contribution that does not vanish because of presence of the boundary. The gauge invariance can be restored if we assume that the apart from the bulk action given by (22), there is a boundary action that couples the gauge field to new boundary degrees of freedom. This boundary action has to possess the property that its gauge transformation cancels the boundary terms (28) and (30) that were obtained from the gauge transformation of the Chern-Simons action (22).

The boundary part will not depend only on the gauge superfield but also on the scalar Lie-algebra valued boundary superfield V. For now, we will assume that the superfield V is defined everywhere, we will see later that it suffices to define it on the boundary. The gauge transformation of this superfield is postulated to be

$$e^{iV}\rightarrow e^{iV}e^{-iK}. \tag{54}$$

It is chosen in this way in order to ensure that the connections $\Gamma_A^{(V)}$, which are finite gauge transformations of Γ_A generated by V, are not changed by gauge transformations $\delta_g^{(K)}\Gamma_A^{(V)}=0$. With the help of $\Gamma_A^{(V)}$ we can write the gauge invariant action as $S^{\mathrm{cs}}[\Gamma_A^{(V)}]$. The invariance of this action follows from the fact that gauge transformations leave $\Gamma_A^{(V)}$ unchanged. The same procedure was also used in [22] for a boundary in a space-like direction.

In the $SIM(1)$ setting we define two superfields corresponding to the Lorentz superfield V

$$v=V|_{\theta_+=0}, \qquad\qquad \nu_-=(D_-V)|_{\theta_+=0}. \tag{55}$$

The gauge invariant action can be written, according to (31) as

$$\begin{aligned}S^{\mathrm{cs}}[\gamma_A^{(V)}]&=S^{\mathrm{cs}}[\gamma_A]+S^{\Delta}[\gamma_A;v,\nu_-]\\&=S^{\mathrm{cs}}_{\mathrm{bulk}}[\gamma_A]+\left(S^{\mathrm{cs}}_{\mathrm{boundary}}[\gamma_A]+S^{\Delta}[\gamma_A;v,\nu_-]\right),\end{aligned} \tag{56}$$

where the expression inside brackets contains all boundary terms. It is important to note that there is no dependence on the superfield v, ν_- in the bulk action, only the surface action $S^{\Delta}[\gamma_A;v,\nu_-]$ (32) depends on these superfields. Furthermore, in order to evaluate

the surface action we need to know only the value of v, ν_- and of their derivatives in directions tangent to the boundary. We do not need to know the derivatives $\partial_{--}v$ or $\partial_{--}\nu_-$ in direction normal to the boundary to evaluate the boundary action. Thus, it is enough if the superfields v, ν_- are defined on the boundary.

In the same way as we defined the superfields k_A, $k_A^{(s)}$ we define the superfields v_A, $v_A^{(s)}$

$$v_A^{(s)} = i(\mathcal{D}_A e^{-isv})e^{isv}, \qquad v_A = v_A^{(s)}|_{s=1}, \qquad v_- = i(D_- e^{-isV})e^{isV}|_{\theta_+=0}. \tag{57}$$

Their infinitesimal gauge transformations are

$$\delta_g^{(K)} v_A = i[k, v_A] - \mathcal{D}_A k, \qquad \delta_g^{(K)} v_- = i[k, v_-] - \kappa_-, \tag{58}$$

The superfields v_A and $v_A^{(s)}$ satisfy the same set of identities (40), (42), (43) and (45) as k_A and $k_A^{(s)}$ with v in place of k. With these definitions we write the boundary part of the action as

$$\begin{aligned} S^{\Delta}[\gamma_A; v, \nu_-] + S^{\rm cs}_{\rm boundary}[\gamma_A] = \frac{k}{4\pi}\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big[-\frac{i}{6}(\gamma_- + v_-)\{\gamma_+ + v_+, \gamma_+ + v_+\} \\ + v_+\gamma_{+-} - \gamma_+ v_{+-} + \int_0^1 \mathrm{d}s \left(\left(\tfrac{\mathrm{d}}{\mathrm{d}s}v_+^{(s)}\right) v_{+-}^{(s)} - v_+^{(s)} \left(\tfrac{\mathrm{d}}{\mathrm{d}s}v_{+-}^{(s)}\right)\right)\Big]. \end{aligned} \tag{59}$$

There is an interesting interpretation for the combination $\gamma_A + v_A$ that appears in the first term. If we replace the ordinary derivatives in the definition of v_A (57) with the covariant ones, then we get

$$v_A^{\nabla} = i(\nabla_A e^{-iv})e^{iv} = i(\mathcal{D}_A e^{-iv} - i\gamma_A e^{-iv})e^{iv} = v_A + \gamma_A. \tag{60}$$

Thus, the first term in (59) can be written as $-\frac{i}{6}(v_-^{\nabla})\{v_+^{\nabla}, v_+^{\nabla}\}$. This term is gauge invariant, we do not need it to restore the gauge invariance, but we need it for $SIM(1)$ invariance.

6 Chern-Simons Theory with Redefined $SIM(1)$ Superfields

In this section, we are going to rewrite the results of the previous sections using $SIM(1)$ superfields that have better $SIM(1)$ transformation properties. The description of the gauge theory in the previous sections was given with the help of a set of superfields γ_+, γ_-, γ_{++}, γ_{+-}, γ_{--}. In this section, we are going to use a different set of superfields consisting of γ_+, $\gamma_\times$, γ_{++}, $\gamma_{\times+}$, $\gamma_{\times\times}$. The superfields γ_+, γ_{++} are defined according to (16) the redefined superfields are defined as [45]

$$\begin{aligned} \gamma_\times &= i\left(\partial_\times{}^\alpha\Gamma_\alpha\right)|_{\theta_+=0} = \gamma_- - \partial_{\times-}\gamma_+, \\ \gamma_{\times+} &= i\left(\partial_\times{}^\alpha\Gamma_{\alpha+}\right)|_{\theta_+=0} = \gamma_{+-} - \partial_{\times-}\gamma_{++}, \\ \gamma_{\times\times} &= -\left(\partial_\times{}^\alpha\partial_\times{}^\beta\Gamma_{\alpha\beta}\right)|_{\theta_+=0} = \gamma_{--} - 2\partial_{\times-}\gamma_{+-} + \partial_{\times-}^2\gamma_{++}. \end{aligned} \tag{61}$$

where the operator $\partial_{\times -}$ is

$$\partial_{\times\alpha} = \frac{\partial_{+\alpha}}{\partial_{++}} \qquad \Leftrightarrow \qquad \partial_{\times +} = 1, \qquad \partial_{\times -} = \frac{\partial_{+-}}{\partial_{++}}. \tag{62}$$

The difference between $SIM(1)$ projections that have been used in the previous section and redefined superfields is that each carry different representations. The $SIM(1)$ projections carry a spinor representation (and its tensor products), while redefined superfields carry representations that we have on $\mathcal{S}_{\text{quotient}}$ and $\mathcal{S}_{\text{invariant}}$ (and their tensor products). If $\times$ is treated as a new type of index, together with $+$ an $-$, then the $SIM(1)$ transformation of any object could be determined by applying the rules

$$\delta_s\psi_+ = A\psi_+, \qquad \delta_s\psi_- = -A\psi_- + B\psi_+, \qquad \delta_s\psi_\times = -A\psi_\times, \tag{63}$$

on each index. If some object has only $+$ and $\times$ indices, which is the case of redefined superfields, then its $SIM(1)$ transformation is especially simple, it can be written as

$$\delta_s\psi_{+\cdots+\times\cdots\times} = A\cdot(\#\text{ of "}+\text{" indices minus }\#\text{ of "}\times\text{" indices})\cdot\psi_{+\cdots+\times\cdots\times}. \tag{64}$$

The infinitesimal gauge transformations of redefined superfields are more complicated that the ones we encountered in the case of $SIM(1)$ projections

$$\begin{aligned}
\delta_g^{(K)}\gamma_\times &= i[k,\gamma_\times] - \partial_\times{}^\alpha[k,\partial_{\times\alpha}\gamma_+] + \kappa_\times, \\
\delta_g^{(K)}\gamma_{\times +} &= i[k,\gamma_{\times +}] - \partial_\times{}^\alpha[k,\partial_{\times\alpha}\gamma_{++}], \\
\delta_g^{(K)}\gamma_{\times\times} &= i[k,\gamma_{\times\times}] - 2\partial_\times{}^\alpha[k,\partial_{\times\alpha}\gamma_{\times +}] - i\partial_\times{}^\alpha\partial_\times{}^\beta[k,\partial_{\times\alpha}\partial_{\times\beta}\gamma_{++}] + \tfrac{\Box}{\partial_{++}}k,
\end{aligned} \tag{65}$$

where $\kappa_\times = \kappa_- - \partial_{\times -}d_+k$.

As before, we write the Chern-Simons action S^{cs} as a sum of a bulk part $S'^{\text{cs}}_{\text{bulk}}$ and a boundary part $S'^{\text{cs}}_{\text{boundary}}$

$$S^{\text{cs}}[\gamma_A] = S'^{\text{cs}}_{\text{bulk}}[\gamma_A] + S'^{\text{cs}}_{\text{boundary}}[\gamma_A]. \tag{66}$$

The prime is used in order to distinguish $S'^{\text{cs}}_{\text{bulk}}$, $S'^{\text{cs}}_{\text{boundary}}$ from the actions $S^{\text{cs}}_{\text{bulk}}$, $S^{\text{cs}}_{\text{boundary}}$. The split of S^{cs} into $S'^{\text{cs}}_{\text{bulk}}$, $S'^{\text{cs}}_{\text{boundary}}$ is not the same as the split into $S^{\text{cs}}_{\text{bulk}}$, $S^{\text{cs}}_{\text{boundary}}$. In fact, we have

$$S'^{\text{cs}}_{\text{boundary}} - S^{\text{cs}}_{\text{boundary}} = -\big(S'^{\text{cs}}_{\text{bulk}} - S^{\text{cs}}_{\text{bulk}}\big) = \frac{k}{4\pi}\text{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big(\frac{i}{6}(\partial_{\times -}\gamma_+)\{\gamma_+,\gamma_+\}\Big). \tag{67}$$

In order to find this result, we have to keep track of ∂_{--} surface terms in the calculation of the action $S'^{\text{cs}}_{\text{bulk}}$ from $S^{\text{cs}}_{\text{bulk}}$ in [45]. The only place where such a surface term appears is the identity

$$\begin{aligned}
&\text{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\Big(-(\partial_{\times -}\gamma_{++})\left(\tfrac{\Box}{\partial_{++}}\gamma_+\right) + \gamma_+\left(\tfrac{\Box}{\partial_{++}}\partial_{\times -}\gamma_{++}\right)\Big) \\
&\qquad = \text{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\Big(-\frac{2}{3}\left(\tfrac{\Box}{\partial_{++}}\gamma_+\right)(\partial_\times{}^\alpha\{\gamma_+,\partial_{\times\alpha}\}) \\
&\qquad\qquad - \frac{2i}{3}\gamma_+[\gamma_{++},\partial_{\times -}^3\gamma_{++}] + \frac{i}{6}\partial_{--}\left((\partial_{\times -}\gamma_+)\{\gamma_+,\gamma_+\}\right)\Big).
\end{aligned} \tag{68}$$

When this identity was derived in [45], the surface term was neglected. We are going to provide a brief description of the proof that keeps track of the mentioned surface term. We make the substitutions $\frac{\Box}{\partial_{++}} = \partial_{--} - \partial_{++}\partial_{\times -}^2$ and $\gamma_{++} = -d_+\gamma_+ + \frac{i}{2}\{\gamma_+, \gamma_+\}$ on the right side of (68), which gives us

$$\begin{aligned}\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\Big((\partial_{\times -}d_+\gamma_+)(\partial_{--}\gamma_+) - \gamma_+(\partial_{--}\partial_{\times -}d_+\gamma_+) - (\partial_{\times -}d_+\gamma_+)(\partial_{++}\partial_{\times -}^2\gamma_+)\\
+ \gamma_+(\partial_{++}\partial_{\times -}^3 d_+\gamma_+) - \frac{i}{2}(\partial_{\times -}\{\gamma_+, \gamma_+\})(\partial_{--}\gamma_+) + \frac{i}{2}\gamma_+(\partial_{--}\partial_{\times -}\{\gamma_+, \gamma_+\})\\
+ \frac{i}{2}(\partial_{\times -}\{\gamma_+, \gamma_+\})(\partial_{++}\partial_{\times -}^2\gamma_+) - \frac{i}{2}\gamma_+(\partial_{++}\partial_{\times -}^3\{\gamma_+, \gamma_+\})\Big). \quad (69)\end{aligned}$$

The first term cancels with the second term, the third term cancels with the fourth term, the rest can be written as

$$\begin{aligned}\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\Big(- \frac{2i}{3}(\partial_{\times -}\{\gamma_+, \gamma_+\})(\partial_{--}\gamma_+) + \frac{i}{3}\gamma_+(\partial_{--}\partial_{\times -}\{\gamma_+, \gamma_+\})\\
+ i(\partial_{++}\partial_{\times -}^3\gamma_+)\{\gamma_+, \gamma_+\} + \frac{i}{6}\partial_{--}\left((\partial_{\times -}\gamma_+)\{\gamma_+, \gamma_+\}\right)\Big). \quad (70)\end{aligned}$$

Notice, that a ∂_{--} surface term appeared as a result of this procedure. The rest of the calculation does not give any other ∂_{--} surface term. The expression (70) can be written as (details can be found in [45])

$$\begin{aligned}\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\Big(- \frac{2}{3}\left((\partial_{--} - \partial_{++}\partial_{\times -}^2)\gamma_+\right)\left(i\partial_{\times -}\{\gamma_+, \gamma_+\} - i\{\gamma_+, \partial_{\times -}\gamma_+\}\right)\\
- \frac{2i}{3}(\partial_{++}\gamma_+)\{\gamma_+, \partial_{\times -}^3\gamma_+\} + \frac{i}{6}\partial_{--}\left((\partial_{\times -}\gamma_+)\{\gamma_+, \gamma_+\}\right)\Big). \quad (71)\end{aligned}$$

This is exactly the expression that is on the right side of (68).

The bulk action has already been calculated in [45]

$$\begin{aligned}S'^{\mathrm{cs}}_{\mathrm{bulk}}[\gamma_A] = \frac{k}{4\pi}\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\Bigg(- 2\gamma_{\times\times}\left(d_+\gamma_{\times +}\right) - \gamma_{\times +}\left(\frac{\Box}{\partial_{++}}\gamma_+\right) + \left(\frac{\Box}{\partial_{++}}\gamma_{\times +}\right)\gamma_+\\
- \frac{2}{3}\left(\frac{\Box}{\partial_{++}}\gamma_+\right)\left(\partial_\times{}^\alpha\{\gamma_+, \partial_{\times\alpha}\gamma_+\}\right) + 2i\gamma_+\left[\gamma_{\times +}, \gamma_{\times\times}\right]\\
- 2\gamma_+\left[\partial_\times{}^\alpha\gamma_{\times +}, \partial_{\times\alpha}\gamma_{\times +}\right] + 2\gamma_+\left[\partial_\times{}^\alpha\gamma_{++}, \partial_{\times\alpha}\gamma_{\times\times}\right]\\
+ 2i\gamma_+\left[\partial_\times{}^\alpha\partial_\times{}^\beta\gamma_{++}, \partial_{\times\alpha}\partial_{\times\beta}\gamma_{\times +}\right] - \frac{1}{3}\gamma_+\left[\partial_\times{}^\alpha\partial_\times{}^\beta\partial_\times{}^\gamma\gamma_{++}, \partial_{\times\alpha}\partial_{\times\beta}\partial_{\times\gamma}\gamma_{++}\right]\Bigg). \quad (72)\end{aligned}$$

The boundary action is obtained by combining the expression for $S^{\mathrm{cs}}_{\mathrm{boundary}}$ (25) with (67), it is given by

$$S'^{\mathrm{cs}}_{\mathrm{boundary}}[\gamma_A] = \frac{k}{4\pi}\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big(- \frac{i}{6}\gamma_\times\{\gamma_+, \gamma_+\}\Big). \quad (73)$$

In this case both the bulk action $S'^{\mathrm{cs}}_{\mathrm{bulk}}$ and the boundary action $S'^{\mathrm{cs}}_{\mathrm{boundary}}$ are $SIM(1)$ invariant. This contrasts with the case of the action (22) where the change of the bulk action $S^{\mathrm{cs}}_{\mathrm{bulk}}$ had to be compensated by the change of the boundary action $S^{\mathrm{cs}}_{\mathrm{boundary}}$.

6.1 Finite Gauge Transformation

We are going to rewrite the expression for the surface term S^{Δ} (32) in terms of redefined superfields. Apart from the dependence on the superfields γ_+, $\gamma_\times$, γ_{++}, $\gamma_{\times+}$, $\gamma_{\times\times}$, which have already been described, there will be a dependence on k_+, $k_\times$, $k_{\times+}$, $k^{(s)}_+$, $k^{(s)}_{\times+}$, where

$$k_\times = k_- - \partial_{\times-}k_+, \qquad k_{\times+} = k_{+-} - \partial_{\times-}k_{++}, \qquad k^{(s)}_{\times+} = k^{(s)}_{+-} - \partial_{\times-}k^{(s)}_{++}. \tag{74}$$

We are going to make the following substitutions

$$\begin{aligned} &\gamma_- = \gamma_\times + \partial_{\times-}\gamma_+, \qquad \gamma_{+-} = \gamma_{\times+} + \partial_{\times-}\gamma_{++} \\ &k_- = k_\times + \partial_{\times-}k_+, \qquad k_{+-} = k_{\times+} + \partial_{\times-}k_{++}, \qquad k^{(s)}_{+-} = k^{(s)}_{\times+} + \partial_{\times-}k^{(s)}_{++}. \end{aligned} \tag{75}$$

The first term in (32) gives

$$\begin{aligned} &\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big[-\frac{i}{6}(\gamma_- + k_-)\{\gamma_+ + k_+, \gamma_+ + k_+\}\Big] \\ &\quad = \mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big[-\frac{i}{6}(\gamma_\times + k_\times)\{\gamma_+ + k_+, \gamma_+ + k_+\} - \frac{i}{6}(\partial_{\times-}\gamma_+)\{\gamma_+, \gamma_+\} \\ &\qquad - \frac{i}{6}(\partial_{\times-}k_+)\{\gamma_+, \gamma_+\} - \frac{i}{3}k_+\{\gamma_+, \partial_{\times-}\gamma_+\} - \frac{i}{6}(\partial_{\times-}\gamma_+)\{k_+, k_+\} \\ &\qquad - \frac{i}{6}\gamma_+\{k_+, \partial_{\times-}k_+\} - \frac{i}{6}(\partial_{\times-}k_+)\{k_+, k_+\}\Big], \end{aligned} \tag{76}$$

the second term gives

$$\begin{aligned} &\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big[\frac{i}{6}\gamma_-\{\gamma_+, \gamma_+\}\Big] \\ &\quad = \mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big[\frac{i}{6}\gamma_\times\{\gamma_+, \gamma_+\} + \frac{i}{6}(\partial_{\times-}\gamma_+)\{\gamma_+, \gamma_+\}\Big]. \end{aligned} \tag{77}$$

The third and the fourth term give

$$\begin{aligned} &\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big[k_+\gamma_{+-} - \gamma_+k_{+-}\Big] \\ &\quad = \mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big[k_+\gamma_{\times+} - \gamma_+k_{\times+} + k_+(\partial_{\times-}\gamma_{++}) - \gamma_+(\partial_{\times-}k_{++})\Big]. \end{aligned} \tag{78}$$

It is convenient to rewrite this expression in such a way that there is no dependence on γ_{++} and k_{++}. We can use (20) and (42) (with $s = 1$) to express γ_{++} and k_{++} by expressions that contain only γ_+, k_+. The result is

$$\begin{aligned} &\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big[k_+\gamma_{\times+} - \gamma_+k_{\times+} - k_+(\partial_{\times-}d_+\gamma_+) + \frac{i}{2}k_+(\partial_{\times-}\{\gamma_+, \gamma_+\}) \\ &\qquad + \gamma_+(\partial_{\times-}d_+k_+) + \frac{i}{2}\gamma_+(\partial_{\times-}\{k_+, k_+\})\Big] \\ &= \mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big[k_+\gamma_{\times+} - \gamma_+k_{\times+} + \frac{i}{2}(\partial_{\times-}k_+)\{\gamma_+, \gamma_+\} + \frac{i}{2}(\partial_{\times-}\gamma_+)\{k_+, k_+\}\Big], \end{aligned} \tag{79}$$

where in the equality we used the fact that if we integrate the third term $-k_+(\partial_{\times -} d_+ \gamma_+)$ by parts to move $\partial_{\times -}$ and d_+ we get $-\gamma_+(\partial_{\times -} d_+ k_+)$, which cancels the fifth term. The terms with the integral over s in (32) give

$$\begin{aligned}
\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\int_0^1 \mathrm{d}s\Big(\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_+^{(s)}\right)k_{+-}^{(s)}-k_+^{(s)}\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_{+-}^{(s)}\right)\Big)& \\
=\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\int_0^1 \mathrm{d}s\Big(\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_+^{(s)}\right)k_{\times +}^{(s)}-k_+^{(s)}\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_{\times +}^{(s)}\right)& \\
+\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_+^{(s)}\right)(\partial_{\times -}k_{++}^{(s)})-k_+^{(s)}\left(\tfrac{\mathrm{d}}{\mathrm{d}s}\partial_{\times -}k_{++}^{(s)}\right)\Big).& \qquad (80)
\end{aligned}$$

With the help of (42), we can write the last two terms only using $k_+^{(s)}$

$$\begin{aligned}
\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\int_0^1 \mathrm{d}s\Big(-\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_+^{(s)}\right)(\partial_{\times -}d_+k_+^{(s)})-\frac{i}{2}\left(\tfrac{\mathrm{d}}{\mathrm{d}s}\partial_{\times -}k_+^{(s)}\right)\{k_+^{(s)},k_+^{(s)}\}& \\
+k_+^{(s)}\left(\tfrac{\mathrm{d}}{\mathrm{d}s}\partial_{\times -}d_+k_+^{(s)}\right)+\frac{i}{2}k_+^{(s)}\left(\tfrac{\mathrm{d}}{\mathrm{d}s}\partial_{\times -}\{k_+^{(s)},k_+^{(s)}\}\right)\Big).& \qquad (81)
\end{aligned}$$

The first and the third term vanish because they can be written as a total d_+ derivative. The second and the fourth term can be written as

$$\begin{aligned}
\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\int_0^1 \mathrm{d}s\Bigg(\frac{i}{6}\frac{\mathrm{d}}{\mathrm{d}s}\left((\partial_{\times -}k_+^{(s)})\{k_+^{(s)},k_+^{(s)}\}\right)& \\
-\frac{2i}{3}\left(\partial_{\times -}\tfrac{\mathrm{d}}{\mathrm{d}s}k_+^{(s)}\right)\{k_+^{(s)},k_+^{(s)}\}+\frac{2i}{3}\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_+^{(s)}\right)\{k_+^{(s)},\partial_{\times -}k_+^{(s)}\}\Bigg),& \qquad (82)
\end{aligned}$$

and this can be written as

$$\begin{aligned}
\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Bigg(\frac{i}{6}(\partial_{\times -}k_+)\{k_+,k_+\}& \\
+\int_0^1 \mathrm{d}s\left[-\frac{2}{3}\left(\partial_\times{}^\alpha\tfrac{\mathrm{d}}{\mathrm{d}s}k_+^{(s)}\right)\{k_+^{(s)},\partial_{\times\alpha}k_+^{(s)}\}\right]\Bigg).& \qquad (83)
\end{aligned}$$

Now, we are going to put together the pieces that we have just calculated. The sum of (76), (77), (79), (83) gives us the action S^Δ written with the help of redefined superfields

$$\begin{aligned}
S'^\Delta[\gamma_A;k,\kappa_\times]=\frac{k}{4\pi}\mathrm{tr}\int \mathrm{d}^3x\mathrm{d}\theta^+\partial_{--}\Big(-\frac{i}{6}(\gamma_\times+k_\times)\{\gamma_++k_+,\gamma_++k_+\}+\frac{i}{6}\gamma_\times\{\gamma_+,\gamma_+\}& \\
+k_+\gamma_{\times +}-\gamma_+k_{\times +}+\frac{1}{3}(\partial_\times{}^\alpha k_+)\{\gamma_+,\partial_{\times\alpha}\gamma_+\})+\frac{1}{3}(\partial_\times{}^\alpha\gamma_+)\{k_+,\partial_{\times\alpha}k_+\})& \\
+\int_0^1 \mathrm{d}s\Big(\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_+^{(s)}\right)k_{\times +}^{(s)}-k_+^{(s)}\left(\tfrac{\mathrm{d}}{\mathrm{d}s}k_{\times +}^{(s)}\right)-\frac{2}{3}\left(\partial_\times{}^\alpha\tfrac{\mathrm{d}}{\mathrm{d}s}k_+^{(s)}\right)\{k_+^{(s)},\partial_{\times\alpha}k_+^{(s)}\}\Big)\Big).& \qquad (84)
\end{aligned}$$

6.2 Boundary Superfield

In section 5.3, we have coupled the gauge superfield to a boundary superfield in order to restore the gauge invariance of the Chern-Simons theory. Now, we are going to reformulate this result with the help of redefined superfields. The gauge invariant action can be written as

$$S^{\rm cs}[\gamma_A^{(V)}] = S'^{\rm cs}_{\rm bulk}[\gamma_A] + \left(S'^{\rm cs}_{\rm boundary}[\gamma_A] + S'^{\Delta}[\gamma_A; v, \nu_\times]\right), \tag{85}$$

where the terms in brackets constitute the boundary part of the action. We should note that both the bulk part and the boundary part are separately $SIM(1)$ invariant. In fact, each of actions $S'^{\rm cs}_{\rm bulk}$, $S'^{\rm cs}_{\rm boundary}$ and S'^{Δ} is separately invariant. This contrasts with the case (56), where $SIM(1)$ projections were used instead of redefined superfields. In that case the parts $S^{\rm cs}_{\rm bulk}$ and $S^{\rm cs}_{\rm boundary}$ were not separately invariant.

The boundary part of the action from (85) is

$$\begin{aligned} S'^{\Delta}[\gamma_A; v, \nu_\times] + S'^{\rm cs}_{\rm boundary}[\gamma_A] = \frac{k}{4\pi}{\rm tr}\int {\rm d}^3x{\rm d}\theta^+\partial_{--}\Big(-\frac{i}{6}(\gamma_\times + v_\times)\{\gamma_+ + v_+, \gamma_+ + v_+\} \\ + v_+\gamma_{\times +} - \gamma_+ v_{\times +} + \frac{1}{3}(\partial_\times{}^\alpha v_+)\{\gamma_+, \partial_{\times\alpha}\gamma_+\}) + \frac{1}{3}(\partial_\times{}^\alpha \gamma_+)\{v_+, \partial_{\times\alpha}v_+\}) \\ + \int_0^1 {\rm d}s\Big(\left(\tfrac{\rm d}{{\rm d}s}v_+^{(s)}\right) v_{\times+}^{(s)} - v_+^{(s)}\left(\tfrac{\rm d}{{\rm d}s}v_{\times+}^{(s)}\right) - \frac{2}{3}\left(\partial_\times{}^\alpha \tfrac{\rm d}{{\rm d}s}v_+^{(s)}\right)\{v_+^{(s)}, \partial_{\times\alpha}v_+^{(s)}\}\Big)\Big), \end{aligned} \tag{86}$$

where

$$v_\times = v_- - \partial_{\times -}v_+, \qquad v_{\times+} = v_{+-} - \partial_{\times-}v_{++}, \qquad v^{(s)}_{\times+} = v^{(s)}_{+-} - \partial_{\times-}v^{(s)}_{++}. \tag{87}$$

7 Conclusion

In this paper, we have analyzed a three dimensional supersymmetric Chern-Simons theory in presence of a boundary. This was done by considering a boundary that satisfied the condition $n \cdot x = 0$, where n is a light-like vector. This boundary was called a light like boundary, and unlike the space-like boundary whose metric was rank two, the metric on this boundary only was rank one. The presence of this boundary broke the symmetry group of the spacetime manifold from the Lorentz group down to the $SIM(1)$ group. Thus, the theory was studied using the $SIM(1)$ superspace. It was demonstrated that this theory only preserved half the supersymmetry of the original theory. As the Chern-Simons theory had $\mathcal{N} = 1$ supersymmetry in absence of a boundary, it only retained $\mathcal{N} = 1/2$ supersymmetry in presence of this boundary. Finally, it was observed that the Chern-Simons theory can be made gauge invariant by introducing new degrees of freedom on the boundary. The gauge transformation of these new degrees of freedom exactly canceled the boundary term obtained from the gauge transformation of the Chern-Simons theory.

The results obtained in this paper could be used to study a system of multiple M2-branes in presence of a boundary in a light-like direction. This would require the coupling of matter fields in the bi-fundamental representation to the Chern-Simons theories. It

may be noted that the coupling of matter fields to Yang-Mills theories has already been studied in fundamental representation [42]. Furthermore, it would also be interesting to generalize this work by considering a Chern-Simons theory in $\mathcal{N} = 2$ superspace formalism. We can analyze the effect of imposing a boundary in the light-like direction for this Chern-Simons theory. It is expected that half of the supersymmetry of the original Chern-Simons theory with $\mathcal{N} = 2$ supersymmetry will be broken in the presence of a boundary. Furthermore, it should also be possible to couple this theory to new degrees of freedom on the boundary such that the resultant theory is gauge invariant. The supersymmetry for an abelian ABJM theory, in presence of a boundary, had also discussed in $\mathcal{N} = 2$ superspace formalism. However, no discussion of supersymmetry of the full non-abelian ABJM theory in $\mathcal{N} = 2$ superspace formalism, or its gauge invariance had been done so far. So, a generalization of these results to $\mathcal{N} = 2$ superspace formalism, and their application to the ABJM theory in presence of a boundary will be interesting. As M2-branes can end on M5-branes, M9-branes or gravitational waves [46], this formalism might be useful to study the physics of such systems.

In order to quantize the action for multiple M2-branes we have to add a gauge fixing term and a ghost term to the original action. The total action thus obtained will be invariant under BRST symmetry [47]-[49]. The BRST symmetry for multiple M2-branes on a manifold without a boundary has been studied in $\mathcal{N} = 1$ superspace formalism [50]-[51]. This analysis has been generalized to include a boundary in a space-like direction [22]. It has been demonstrated that the bulk action for multiple M2-branes is not invariant under the BRST transformations. The BRST transformations for this action generate a boundary contribution. However, the BRST transformations of the new boundary degrees of freedom exactly cancel the boundary contribution generated from the BRST transformation of the bulk action for M2-branes. It will be interesting to investigate this for a boundary in the light-like direction.

It has been demonstrated that using the Hořava-Witten theory, one of the low energy limits of the heterotic string theory can be obtained from the eleven dimensional supergravity in presence of a boundary [31]-[32]. Thus, the strong-coupling limit of the type IIA string theory has been related to the strong-coupling limit of the heterotic string [52]-[53]. This was done by compactifying the original theory on an interval bounded by mirror orientifold planes. It was argued that a ten dimensional E_8 super-Yang-Mills theory appears on each plane. So, two E_8 gauge theories were obtained on the mirror planes, and supergravity was obtained between these planes. In this construction, the low-energy value of the Newton's constant decreases when the distance between the planes is increased. The gauge coupling remains fixed as this interval is increased. Thus, by adjustment of the length of this interval, it was possible to obtain a unification of gauge and gravitational couplings. In this theory, six dimensions were compactified, and thus, a five dimensional theory on an interval with mirror plane boundaries was obtained. This theory is expected to be a five dimensional supergravity model, with additional bulk super-multiplets. It has been argued that the analysis of a simpler system can help understand the Hořava-Witten theory. A simplified construction of a five dimensional globally supersymmetric Yang-Mills theory coupled to a four dimensional hypermultiplet on the

boundary has also been analyzed [54]. It would be interesting to analyze what features of this model can be retained for a boundary in a light like direction. Thus, it will be interesting to generalize the results of this paper to five dimensions and use it for analyzing a globally supersymmetric Yang-Mills theory.

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